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MEASURE OF MAXIMAL ENTROPY FOR STAR MULTIMODAL MAPS

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Abstract. Let $f : [0,1] \to [0,1]$ be a multimodal map with positive topological entropy. The dynamics of the renormalization operator for multimodal maps have been investigated by Daniel Smania. It is proved that the measure of maximal entropy for a specific category of C^r interval maps is unique.

1. Introduction

A map $f : [0,1] \to [0,1]$ is called a multimodal map, if f is a smooth map with a finite number of critical points, all of them local maximum or local minimum. The dynamics of the renormalization operator for multimodal maps have been investigated by Daniel Smania [10]. The uniqueness of the measure of maximal entropy has been investigated in several previous works in different settings([3], [7], [2]). The uniqueness of the measure of maximal entropy for unimodal maps with positive entropy has been proved by Hofbauer in [3]. A simpler problem is presented by Raith in [7]. Hofbauer investigated intrinsic ergodicity of topologically transitive dynamical systems [3]. In this paper, we investigate the following questions:

Question A.whether the measure $\mu(f)$ of maximal entropy for the multimodal map is unique; or what conditions are necessary?

A map $f : [0,1] \to [0,1]$ is called a unimodal map, if f is continuous and there is a $c \in (0,1)$ such that f is strictly increasing on $[0,c]$ and f

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is strictly decreasing on $[c, 1]$ (or f is strictly decreasing on $[0, c]$ and f is strictly increasing on $[c, 1]$).

Question B. Let $f_1, \ldots, f_n$ are unimodal maps, with unique measures $\mu_1(f_1), \ldots, \mu_n(f_n)$ of maximal entropy, respectively, and assume that $f := f_n \circ \ldots \circ f_1$. what is the relationship between the number of measure $\mu(f)$ of maximal entropy and the number of $\mu_i(f_i)$?

In [3] Hofbauer has studied the interval maps with finite critical sets, and has shown that, for these maps, the set of measures of maximal entropy is never empty. In this paper by rigorously building up the proof, we answer the question A for star multimodal map. In fact we study more specific kinds of multimodal maps, say star multimodal map, and we show that the measure of maximal entropy is unique for star multimodal map. We don't have any answer for question B.

Definition 1.1. **(Star Condition)**: Let f be a multimodal map. There exists $0 = a_0 < a_1 < a_2 < a_3 < ... < a_{n-1} < a_n = 1$ such that f is strictly monotone on which $[a_{i-1}, a_i]$ and $f'(a_i) = 0$ for any $1 \leq i \leq n-1$, and also satisfies the following conditions:

$$f(a_0) = a_0,$$

$$f(a_1) = a_2, \quad f(a_2) = a_1,$$

$$f(a_3) = a_4, \quad f(a_4) = a_3,$$

$$\vdots$$

$$f(a_n) = \begin{cases} a_n & \text{if } n \text{ is an odd number} \\ a_{n-1} & \text{if } n \text{ is an even number} \end{cases}$$

Definition 1.2. A multimodal map f is called *star multimodal map* if satisfying the star condition.

In this paper we prove that the measure $\mu(f)$ of maximal entropy is unique for star multimodal map f. By finiteness and connectivity of Buzzi- Hofbauer graph, this leads us to our main propose:

THEOREM 1.3 (Main Theorem). *Let $f \in C^r([0,1])$ be a star multimodal map. Then the measure $\mu(f)$ of maximal entropy is unique.*

To this end, we prove the finiteness of the Buzzi-Hofbauer diagrams associated to these maps and then, by the results obtained from [5], the final result is the maximal measure of entropy is unique.The proof of the main theorem is based on the study of the Buzzi-Hofbauer diagram and its behavior under C^r perturbations. This diagram is introduced in section 2. In section 3, we investigate the finiteness of the Buzzi-Hofbauer diagram corresponding to the multimodal map which has the conditions mentioned in this paper. Finally, according to a theorem of Burguet in [1], the mapping in this paper has a unique measure of maximal entropy.

2. Buzzi-Hofbauer diagrams

Let f be a star multimodal map with the following properties:

There exists $0 = a_0 < a_1 < a_2 < ... < a_{n-1} < a_n = 1$ such that f is strictly monotone on which $[a_{i-1}, a_i]$, and $f'(a_i) = 0$ for any $1 \leq i \leq n-1$.

Let $C(f)$ be the set of critical points of f. In order to describe the construction of the Buzzi-Hofbauer diagram, we introduce the following definitions from [1]. The construction of Buzzi-Hofbauer diagrams involves the use of follower sets.

DEFINITION 2.1. A countable collection P of the connected components of $[0,1] - C(f)$ is called a *natural partition* of f.

For a natural partition P of f the two-sided symbolic dynamic $\Sigma(f, P)$ associated to f is defined as the shift on the closure in $P^{\mathbb{Z}}$ (for the product topology) of the two-sided sequences $A = (A_n)_n$ such that for all $n \in \mathbb{Z}$ and $l \in \mathbb{N}$ the word $A_n \dots A_{n+l}$ is *admissible*, by which we mean that the intersection $\bigcap_{k=0}^{l} f^{-k}(A_{n+k})$ is nonempty and the f^{l+1} - image of this open interval is not reduced to a point.

REMARK 2.2. Notice that the set $A := \bigcap_{k=0}^{l} f^{-k}(A_{n+k})$ is an open interval. It is enough to show that A is an interval. Suppose that for $x, z \in A$ there is $x < y < z$ such that $y \notin A$. Since $x, z \in A_n$ and A_n is an interval we have $y \in A_n$. In the other hand we have $f(x)$ and $f(z)$ are in A_{n+1}. Since f is strictly monotone on the sets A_i, then $f(y)$ is between $f(x)$, $f(z)$. Therefore $f(y) \in A_{n+1}$. By induction $f^k(y) \in A_{n+k}$, for any $k = 0, \dots, l$. So $y \in A$.

Notice that A consists of finite intersection of open sets.

Definition 2.3. The *follower set* of a finite P-word $B_n \dots B_{n+l}$ is $fol(B_n \dots B_{n+l}) = \{A_{n+l}A_{n+l+1} \dots \in P^{\mathbb{N}}, \quad s.t. \quad \text{there exists} \quad (A_n) \in \Sigma(f,P) \quad with \quad A_n \dots A_{n+l} = B_n \dots B_{n+l}\}$.

In other words, the follower set of a finite word (block) $B_n \dots B_{n+l}$ consists of all one sided rays $A_{n+l}A_{n+l+1} \dots$ which have an extension to a two-sided sequence $A = \dots A_nA_{n+1} \dots A_{n+l}A_{n+l+1} \dots$ such that $A_n \dots A_{n+l} = B_n \dots B_{n+l}$.

Let $\mathcal{P}$ be the set of admissible P-words. We consider the following equivalence relation on $\mathcal{P}$. We say $A_{-n} \dots A_0 \sim B_{-m} \dots B_0$ if and only if there exist $0 \le k \le min(m,n)$ such that

- $A_{-k} \dots A_0 = B_{-k} \dots B_0$
- $fol(A_{-n} \dots A_0) = fol(A_{-k} \dots A_0)$
- $fol(B_{-m} \dots B_0) = fol(B_{-k} \dots B_0)$

The Buzzi-Hofbauer diagram is defined as an oriented graph whose vertices are elements of $\mathcal{D} = \mathcal{D}(f,P) := \mathcal{P}/\sim$ and whose arrows are defined by $\alpha \to \beta, (\alpha, \beta \in \mathcal{D})$ if and only if there exists an integer n and $A_{-n} \dots A_0A_1 \in \mathcal{P}$ such that $\alpha \sim A_{-n} \dots A_0$ and $\beta \sim A_{-n} \dots A_0A_1$.

Therefore, according to the above definitions, for each map f, which has the conditions mentioned in this paper, there is an oriented graph known as the Buzzi-Hofbauer diagram.

It is a well-known result of Shanon and Parry ([5],[8]) that every irreducible subshift of finite type on a finite alphabet has a unique measure μ of maximal entropy for the shift transformation. So a finite connected graph admits a unique measure of maximal entropy (the so called Parry measure).

Therefore, if the finiteness and connectivity of the graph corresponding to f are proved, existence and uniqueness of the maximal measure of entropy are derived. For this purpose we will go on to prove the main theorem in the next section.

3. Proof of main theorem

We can now prove the main theorem. Let f be a star multimodal map, and $A_i := (a_{i-1}, a_i)$ for each $i = 1, 2, \dots, n$.

First, notice some of the properties of f in the following proposition:

Proposition 3.1. *Let f satisfying star condition. Then the set of admissible P-words will be as follows:*

(i) *If the word ends with* A_{odd}, A_{2k-1}, *the only acceptable case will be a constant sequence.*

(ii) *If the word ends with* A_{even}, A_{2k}, *three things may happen:*
The first case. $A_{2k-1}A_{2k-1}\dots A_{2k-1}A_{2k}A_{2k}\dots A_{2k}$.
The second case. $A_{2k+1}A_{2k+1}\dots A_{2k+1}A_{2k}A_{2k}\dots A_{2k}$.
The third case. $A_{2k}A_{2k}\dots A_{2k}$.

Proof. This proposition is based on the following properties:

- The first property. If in a place of the sequence, the phrase A_{2k} will appear, then by star condition the rest of the sequence will be A_{2k} to the end.

  Because if x is in $A_{2k} = (a_{2k-1}, a_{2k})$, $k = 1, 2, \dots$, then $f(x)$ falls on A_{2k}, and similarly all iterates of f will fall on interval A_{2k}.

  So, given the definition of admissible words, if in a place of the sequence, the phrase A_{2k} will appear, then the rest of the sequence will only be A_{2k} to the end.
- The second property. In no part of the sequence, before $A_{2k-1}, (k = 1, 2, \dots)$, $A_{2k+1}, A_{2k+3}, A_{2k+5}, A_{2k+7}, \dots$, can not appear at all.

  Because if x is in $A_{2k-1+2i}, (i = 1, 2, \dots)$, then by star condition $a_{(2k-1+2i)-2} \le f(x) \le a_{(2k-1+2i)+1}$ in the sense that $x \in A_{2k-1+2i}$ yields $f(x) \in$ union of three intervals, that's mean $f(x) \in A_{(2k-1+2i)-1} \cup A_{2k-1+2i} \cup A_{(2k-1+2i)+1}$, so $f(x), f^2(x), f^3(x), \dots$ will not fall on A_{2k-1} in any way.

  Therefore, before $A_{2k-1}, (k = 1, 2, \dots)$, none of $A_{2k+1}, A_{2k+3}, A_{2k+5}, \dots$ will appear.
- The third property. At no point in the sequence, A_{2k-1} never comes before $A_{2k+1}, A_{2k+3}, A_{2k+5}, \dots$.

  Because if x is in $A_{2k-1}, (k = 1, 2, \dots)$, then by star condition $f(x) \in A_{2k-2} \cup A_{2k-1} \cup A_{2k}$. So $f(x), f^2(x), \dots$ will not fall into any of $A_{2k+1}, A_{2k+3}, A_{2k+5}, \dots$. So the result is established.
- The fourth property. Before A_{2k}, only $A_{2k-1}, A_{2k}, A_{2k+1}$ can appear.

  Because if x belongs to A_{2k-1}, then by star condition $f(x) \in A_{2k-2} \cup A_{2k-1} \cup A_{2k}$. So $f(x)$ can be a member of A_{2k}. On the other hand, if x belongs to a member of partition other than A_{2k-1}, A_{2k}, and A_{2k+1}, then $f(x)$ has no relation to A_{2k}, and $f(x)$ will not be in any way in A_{2k}.

For convinience we prove (i) by using the above properties. Similarly one can prove (ii).

Proof of (i) Consider P-word ends with A_{odd} (A_{2k-1}). By the definition of this admissible P-word $A_n, \ldots A_{n+l}$, the set $\bigcap_{k=0}^{l} f^{-k}(A_{n+k})$ is nonempty. By the first property we show that there is no A_{even} before A_{2k-1}. Suppose that there is $A_{2r} = [a_{2r-1}, a_{2r}]$ in some place before A_{2k-1}. By star condition, the images of f are contained in $A_{2r} = [a_{2r-1}, a_{2r}]$, which is contradiction with definition of admissible word . Also, following the second and fourth properties there is no $A_{2k'-1} = [a_{2k'-2}, a_{2k'-1}]$ before A_{2k-1}. By star condition, for $x \in [a_{2k'-2}, a_{2k'-1}]$, $f(x) \in A_{2k'-2} \cup A_{2k'-1} \cup A_{2k'}$ and $f^2(x) \in A_{2k'-1}$. Hence the images $f(x), f^2(x), \ldots$ can not be in A_{2k-1} which contradicts with definition of admissible words.

$\square$

End of the proof of main theorem

For mappings presented in this paper, the admissible words can be in several ways of the following:

1. The words that end with A_1.

Such words will be a constant sequence of A_1 with respect to the features mentioned above.

2. The words that end with A_2.

According to the fourth property, such words will be in the following two forms:

- A number of symbols A_1 and then some A_2 symbols to the end.
- A constant sequence consisting of only A_2.

3. The words that end with A_3.

Given the properties mentioned above, such words will only be in the form of a constant sequence consisting of A_3.

4. The words that end with A_4.

According to the fourth property, such words will be in the following three forms:

- A number of symbols A_3 and then some A_4 symbols to the end.
- A number of symbols A_5 and then some A_4 symbols to the end.
- A constant sequence consisting of only A_4.

5.The words that end with A_5.

Similar to case 3.

$\vdots$

So if the word ends with A_{odd}, the only acceptable case will be a constant sequence, and if the word ends with A_{2k}, three things may happen:

The first case. $A_{2k-1}A_{2k-1}\dots A_{2k-1}A_{2k}A_{2k}\dots A_{2k}$.

The second case. $A_{2k+1}A_{2k+1}\dots A_{2k+1}A_{2k}A_{2k}\dots A_{2k}$.

The third case. $A_{2k}A_{2k}\dots A_{2k}$.

Now we need to identify equivalence classes to complete graph construction.

The words ending in A_{odd} are all equivalent. For example suppose we want to prove $\underbrace{A_{2k+1}\dots A_{2k+1}}_{\text{l repetitions}} \sim \underbrace{A_{2k+1}\dots A_{2k+1}}_{\text{r repetitions}}$. For this purpose, we must prove $fol(\underbrace{A_{2k+1}\dots A_{2k+1}}_{\text{l repetitions}}) = fol(\underbrace{A_{2k+1}\dots A_{2k+1}}_{\text{r repetitions}})$, that the two sets are clearly equal.

Similarly, with some computation, it follows that the resulting graph has n vertices and $2n-1$ edges, which is a finite connected graph and therefore desired result will be proved.

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